MONGE-AMPÈRE MEASURES FOR CONVEX BODIES AND BERNSTEIN-MARKOV TYPE INEQUALITIES

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Abstract. We use geometric methods to calculate a formula for the complex Monge-Ampère measure $(dd^cV_K)^n$, for $K \Subset \mathbb{R}^n \subset \mathbb{C}^n$ a convex body and V_K its Siciak-Zaharjuta extremal function. Bedford and Taylor had computed this for symmetric convex bodies K. We apply this to show that two methods for deriving Bernstein-Markov-type inequalities, i.e., pointwise estimates of gradients of polynomials, yield the same results for all convex bodies. A key role is played by the geometric result that the extremal inscribed ellipses appearing in approximation theory are the maximal area ellipses determining the complex Monge-Ampère solution V_K.

1. Introduction.

For a function u of class C^2 on a domain $\Omega \subset \mathbb{C}^n$, the complex Monge-Ampère operator applied to u is

$$(dd^cu)^n := i\partial\overline{\partial}u \wedge \cdots \wedge i\partial\overline{\partial}u.$$

For a plurisubharmonic (psh) function u which is only locally bounded, $(dd^cu)^n$ is well-defined as a positive measure. Given a bounded set $E \subset \mathbb{C}^n$, we define the Siciak-Zaharjuta extremal function

$$V_E(z) := \sup\{u(z) : u \in L(\mathbb{C}^n),\ u \leq 0 \text{ on } E\}$$

where $L(\mathbb{C}^n)$ denotes the class of psh functions u on $\mathbb{C}^n$ with $u(z) \leq \log^+|z| + c(u)$. If E is non-pluripolar, the upper-regularized function

$$V_E^*(z) := \limsup_{\zeta\to z} V_E(\zeta)$$

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is a locally bounded psh function which satisfies $(dd^cV_E^*)^n = 0$ outside of $\overline{E}$ and the total mass of $(dd^cV_E^*)^n$ is $(2\pi)^n$.

In this paper we consider $E = K \subset \mathbb{R}^n$ a convex body, that is, a compact, convex set with non-empty interior. In this situation the function $V_K = V_K^*$ is continuous but it is not necessarily smooth, even if K is smoothly bounded and strictly convex. Indeed, for $K = \mathbb{B}_{\mathbb{R}}^n$, the unit ball in $\mathbb{R}^n \subset \mathbb{C}^n$, Lundin found [9], [1] that

$$V_K(z) = \frac{1}{2}\log h(|z|^2 + |z\cdot z - 1|), \tag{1.1}$$

where $|z|^2 = \sum |z_j|^2$, $z \cdot z = \sum z_j^2$, and $h(\frac{1}{2}(t + \frac{1}{t})) = t$, for $1 \leq t \in \mathbb{R}$. In this example, the Monge-Ampère measure $(dd^cV_K)^n$ has the explicit form

$$(dd^cV_K)^n = n!\mathrm{vol}(K)\frac{dx}{(1-|x|^2)^{\frac{1}{2}}} := n!\mathrm{vol}(K)\frac{dx_1 \wedge \cdots \wedge dx_n}{(1-|x|^2)^{\frac{1}{2}}}.$$

The main result of this paper is a general formula for this measure (see Theorem 4.1 and Corollary 4.5).

Theorem 1.1. *Let K be a convex body and V_K its Siciak-Zaharjuta extremal function. The limit*

$$\delta_B^K(x,y) = \delta_B(x,y) := \lim_{t\to 0^+}\frac{V_K(x+ity)}{t} \tag{1.2}$$

exists for each $x \in K^o$ and $y \in \mathbb{R}^n$ and for $x \in K^o$

$$(dd^cV_K)^n = \lambda(x)dx \text{ where } \lambda(x) = n!\mathrm{vol}(\{y : \delta_B(x,y) \leq 1\}^*). \tag{1.3}$$

Moreover, $(dd^cV_K)^n$ puts no mass on the boundary ∂K (relative to $\mathbb{R}^n$).

Here, for a symmetric convex body E in $\mathbb{R}^n$,

$$E^* := \{y \in \mathbb{R}^n : x\cdot y \leq 1, \text{ for all } x \in E\}$$

is also a symmetric convex body in $\mathbb{R}^n$, called the polar of E. The quantity $\delta_B(x,y)$ is continuous on $K^o \times \mathbb{R}^n$ and for each fixed $x \in K^o$, $y \to \delta_B(x,y)$ is a norm on $\mathbb{R}^n$; i.e., $\delta_B(x,y) \geq 0$; $\delta_B(x,\lambda y) = \lambda\delta_B(x,y)$ for $\lambda \geq 0$; and $\delta_B(x,y_1+y_2) \leq \delta_B(x,y_1) + \delta_B(x,y_2)$ (see [3]).

For a symmetric convex body, i.e., $K = -K$, Bedford and Taylor [3] showed the existence of the limit (1.2) and proved the formula (1.3) using the description of the Monge-Ampère solution given by Lundin [10]. The present paper relies on the description of V_K given in [6], [7] for general convex bodies K. [6] showed the existence, through each point $z \in \mathbb{C}^n \setminus K$, of a holomorphic curve on which V_K is harmonic,

while [7] showed that for many K (all K in $\mathbb{R}^2$) these curves give a continuous foliation of $\mathbb{C}^n \setminus K$ by holomorphic curves. It also showed that these curves are algebraic curves of degree 2, and interpreted them in terms of a (finite dimensional) variational problem among real ellipses contained in K. The real points of such a quadratic curve describe an ellipse within K of maximal area in its class of competitors. These competitor classes are specified by the points c on the hyperplane H at infinity in $\mathbb{P}^n$ through which the quadratic curves pass. The geometry of these foliations is our main tool.

The norm $\delta_B(x, y)$ is also related to Bernstein-Markov inequalities for real, multivariate polynomials on K. This will be explained in section 3, specifically, in equation (3.2) and the remarks after it. Conversely, a key role in the proof of the main result Theorem 1.1 is played by the observation (Proposition 3.2) that the extremal inscribed ellipses appearing in a geometric approach – the "inscribed ellipse method" of Sarantopoulos [14], [13] – to Bernstein-Markov inequalities are the maximal area ellipses appearing in the determination of the Monge-Ampère solution as described above. A corollary of our main result is that the inscribed ellipse method and the pluripotential-theoretic method, due to Baran [1, 2] for obtaining Bernstein-Markov-type estimates are equivalent for all convex bodies. This was straightforward for symmetric convex bodies. It was proved for simplices and conjectured for the general case as "Hypothesis A" in [12].

The remainder of the paper is organized as follows. In section 2 we recall in more detail some of the features of the leaf structure for the Monge-Ampère foliation. In section 3 we review the maximal inscribed ellipse problem from [14], [13], its relation to Bernstein-Markov inequalities from approximation theory and to the Monge-Ampère maximal ellipses in section 2. We also sketch the relation to the extremal function V_K for symmetric convex bodies [3], [2]. Finally, in section 4, using details of the Monge-Ampère foliation and its continuity, we prove the main results.

2. Review of the variational problem.

Let $K \subset \mathbb{R}^n \subset \mathbb{C}^n$ be a convex body, and consider $\mathbb{C}^n \subset \mathbb{P}^n$, the complex projective space with $H := \mathbb{P}^n \setminus \mathbb{C}^n$ the hyperplane at infinity. Let $\sigma : \mathbb{P}^n \to \mathbb{P}^n$ be the anti-holomorphic map of complex conjugation, which preserves $\mathbb{C}^n$ and H, and is the identity on $\mathbb{R}^n$. Let $H_{\mathbb{R}}$ denote the

real points of H (fixed points of σ in H). For any non-zero vector $c \in \mathbb{C}^n$, let $\sigma(c) = \overline{c}$, and $[c] \in H$ the point in H given by the direction of c. If $[c] \neq [\overline{c}]$, then $c, \overline{c}$ span a complex subspace $V \subset \mathbb{C}^n$ of dimension two, which is real, that is, invariant under σ; hence V is the complexification of a two-dimensional real subspace $V_0 \subset \mathbb{R}^n$. If we translate V by a vector $A \in \mathbb{R}^n$, we get a complex affine plane $V + A$ invariant by σ and containing the real form $V_0 + A$, the fixed points of σ in $V + A$. Associated to the point $[c] \in H$, we consider holomorphic maps $f : \triangle \to \mathbb{P}^n$, $\triangle$ the unit disk in $\mathbb{C}$, such that $f(0) = [c]$, and $f(\partial\triangle) \subset K$. Such maps can be extended by Schwarz reflection to maps (still denoted) $f : \mathbb{P}^1 \to \mathbb{P}^n$ by the formula

$$f(\tau(\zeta)) = \sigma(f(\zeta)) \in \mathbb{P}^n \tag{2.1}$$

where $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is the inversion $\tau(\zeta) = 1/\overline{\zeta}$. In particular, such maps have the form

$$f(\zeta) = \rho\frac{C}{\zeta} + A + \rho\overline{C}\zeta, \tag{2.2}$$

where $[c] = [C]$, i.e., $C = \lambda c$, for some $\lambda \in \mathbb{C}$, $A \in \mathbb{R}^n$, and $\rho > 0$. Then $f(\mathbb{P}^1) \subset \mathbb{P}^n$ is a quadratic curve, and restricted to $\partial\triangle$, the unit circle in $\mathbb{C}$, f gives a parametrization of a real ellipse inside the planar convex set $K \cap \{V_0 + A\}$, with center at A. According to [7], the extremal function V_K is harmonic on the holomorphic curve $f(\triangle \setminus \{0\}) \subset \mathbb{C}^n \setminus K$ if and only if the area of the ellipse bounded by $f(\partial\triangle)$ is *maximal* among all those of the form (2.2).

For a fixed, normalized C, this is equivalent to varying $A \in \mathbb{R}^n$ and $\rho > 0$ among the maps in (2.2) with $\mathcal{E} = f(\partial\triangle) \subset K$ in order to maximize ρ. Fixing C amounts to prescribing the orientation (major and minor axis) and eccentricity of a family of inscribed ellipses in K. We will call an extremal ellipse $\mathcal{E}$ a *maximal area ellipse*, or simply $a-$*maximal*. In the case where ∂K contains no parallel faces, for each $[c] \in H$ there is a unique $a-$maximal ellipse (Theorem 7.1, [7]); we denote the corresponding map by f_c. In this situation, the collection of complex ellipses $\{f_c(\triangle \setminus \{0\}) : [c] \in H\}$ form a continuous foliation of $\mathbb{C}^n \setminus K$. In simple terms, this means that if z, z' are distinct points in $\mathbb{C}^n \setminus K$, with $|z - z'|$ small, lying on leaves $L(z) := f_c(\triangle \setminus \{0\})$ and $L'(z) := f_{c'}(\triangle \setminus \{0\})$, then the corresponding leaf parameters in (2.2) are close; i.e., $C \sim C'$, $\rho \sim \rho'$ and $A \sim A'$ (and of course $|\zeta_z| \sim |\zeta_{z'}|$ where $f_c(\zeta_z) = z$ and $f_{c'}(\zeta_{z'}) = z'$). Any convex body in $\mathbb{R}^2$ admits a

continuous foliation; this follows from Proposition 9.2 or Theorem 10.2 in [7]. Moreover, if we let $\mathcal{C}$ denote the set of all convex bodies $K \subset \mathbb{R}^n$ admitting a continuous foliation, then $\mathcal{C}$ is dense in the Hausdorff metric in the set $\mathcal{K}$ of all convex bodies $K \subset \mathbb{R}^n$. This follows, for example, from the fact that strictly convex bodies K belong to $\mathcal{C}$ (cf., Theorem 7.1 of [7]). In addition, all symmetric convex bodies admit a continuous foliation.

For convenience, instead of using the holomorphic curves $f(\triangle \setminus \{0\})$ we will work with the holomorphic curve $f(\mathbb{C} \setminus \overline{\triangle})$; thus V_K being harmonic on this curve means that

$$V_K(f(\zeta)) = \log|\zeta| \text{ for } |\zeta| \geq 1. \tag{2.3}$$

3. Inscribed ellipse problem.

Let $K \subset \mathbb{R}^n$ be a convex body. Consider the following geometric problem: fix $x \in K^o$ and a non-zero vector $y \in \mathbb{R}^n$ and consider all ellipses $\mathcal{E}$ lying in K which contain x and have a tangent at x in the direction y. We write $y \in T_x\mathcal{E}$. That is, $\mathcal{E} = \mathcal{E}_b = \mathcal{E}_b(x, y)$ is given by a parameterization

$$\theta \to r(\theta) := a\cos\theta + by\sin\theta + (x - a) \tag{3.1}$$

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^+$ are such that $r(\theta) \in K$ for all θ. *The problem is to maximize b among all such ellipses.* We will call such an ellipse a *maximal inscribed ellipse* (for x, y) or simply $b-$*maximal*. Note that $r(0) = x$ and $r'(0) = by$; thus one is allowed to vary a and b in (3.1). Often we will normalize and assume that y is a unit vector. An observation which will be used later is that if we fix a, then $\mathcal{E}_b$ lies "inside" $\mathcal{E}_{b'}$ if $b < b'$ with two common points x and $x - 2a$.

We give some motivation for studying this problem; this goes back to Sarantapolous (cf., [14] or [13]). For any such ellipse $\mathcal{E}$, if p is a polynomial of n real variables of degree d, say, with $||p||_K \leq 1$, then $t(\theta) := p(r(\theta))$ is a trigonometric polynomial of degree at most d with $||t||_{[0,2\pi]} \leq ||p||_K \leq 1$ (since $\mathcal{E} \subset K$). By the Bernstein-Szegö inequality for trigonometric polynomials,

$$\frac{|t'(\theta)|}{\sqrt{||t||^2_{[0,2\pi]} - t(\theta)^2}} \leq d.$$

From the chain rule,

$$|t'(0)| = |\nabla p(x) \cdot r'(0)| = |D_{by}p(x)| = b|D_y p(x)|.$$

Thus

$$b|D_y p(x)| = |t'(0)| \leq d\sqrt{||t||^2_{[0,2\pi]} - t(0)^2} \leq d\sqrt{1 - p(x)^2};$$

i.e.,

$$\frac{1}{\deg p} \frac{|D_y p(x)|}{\sqrt{1 - p(x)^2}} \leq \frac{1}{b}. \tag{3.2}$$

The left-hand-side is related to what we shall refer to as a *Bernstein-Markov inequality*[1]: it relates the directional derivative of p at x in the direction y with the sup-norm of p on K (the "1" on the right-hand-side of (3.2)) and the degree of p. This motivates the definition of the *Bernstein-Markov pseudometric* (cf., [5]): given $x \in K$, $y \in \mathbb{R}^n$, let

$$\delta_M^K(x; y) = \delta_M(x; y) := \sup_{||p||_K \leq 1,\ \deg p \geq 1} \frac{1}{\deg p} \frac{|D_y p(x)|}{\sqrt{1 - p(x)^2}}.$$

(This definition makes sense for general compacta in $\mathbb{R}^n$). Inequality (3.2) says that *whenever you have an inscribed ellipse $\mathcal{E}_b = \mathcal{E}_b(x, y)$ through x with tangent at x in the direction of y, the number $1/b$ gives an upper bound on the Bernstein-Markov pseudometric*:

$$\delta_M(x; y) \leq 1/b.$$

The bigger you can make b, the better estimate you have.

Let

$$b^*(x, y) := \sup\{b : \mathcal{E}_b(x, y) \subset K\}. \tag{3.3}$$

Note that $b^*(x, ty) = b^*(x, y)/t$ for $t > 0$. In the symmetric case, this is intimately related to V_K:

Proposition 3.1. *If $K = -K$, then $\delta_M(x; y) = \frac{1}{b^*(x,y)}$. Moreover,*

$$\delta_M(x; y) = \lim_{t \to 0^+} \frac{V_K(x + ity)}{t}.$$

[1]For any univariate algebraic polynomial of degree not exceeding n, the sharp uniform estimate for the derivative $\|p'\|_{\infty,[-1,1]} \leq n\|p\|_{\infty,[-1,1]}$ is due to Markov, while the pointwise estimate $|p'(x)|\sqrt{1 - x^2} \leq n\|p\|_{\infty,[-1,1]}$ is known as Bernstein's Inequality, see e.g. [4], pages 232-233. In approximation theory, these types of derivative estimates – or, in the multivariate case, gradient and directional derivative estimates – are usually termed Bernstein and/or Markov type inequalities.

As in the introduction, define

$$\delta_B^K(x;y) = \delta_B(x,y) := \lim_{t\to 0^+} \frac{V_K(x+ity)}{t}, \tag{3.4}$$

provided this limit exists. For symmetric convex bodies K, the proposition says that the limit does exist and we have

$$\delta_B(x,y) = \delta_M(x,y) = \frac{1}{b^*(x,y)}. \tag{3.5}$$

Moreover, for each fixed $x \in K^o$, the function $y \to \frac{1}{b^*(x,y)}$ is a norm (cf., Proposition 3.3). A proof of the existence of the limit was given by Bedford and Taylor [3]. We sketch an alternate proof due to Baran [2].

Step 1: $V_K(z) = \sup\{\log|h(z \cdot Z))| : Z \in K^*\}$ where $h(w) = w + \sqrt{w^2-1}$ is the standard Joukowski map and

$$K^* := \{Z : x \cdot Z \leq 1 \text{ for all } x \in K\}$$

is the polar of K (cf., [10], or [2], Proposition 1.15).

Step 2: We have the following explicit estimates on h: if $|\alpha| < 1$, $|\beta| \leq \sqrt{1-|\alpha|}$, and $0 < \epsilon \leq 1/2$, then

$$(1-\epsilon)\frac{|\beta|}{\sqrt{1-\alpha^2}} \leq \frac{1}{\epsilon}\log|h(\alpha + i\epsilon\beta)| \leq \frac{|\beta|}{\sqrt{1-\alpha^2}}$$

(the inequality on the right-hand-side is valid without the restriction $|\beta| \leq \sqrt{1-|\alpha|}$; cf., [2], Proposition 1.13). This states precisely that $\log|h|$ is Lipschitz as you approach $(-1,1)$ vertically and the Lipschitz constant grows like one-over-the-distance to the boundary points.

Now fix $x \in K^o$ and $y \in \mathbb{R}^n$; then for any $Z \in K^*$ and for $t > 0$ small, since $(x+ity)\cdot Z = x \cdot Z + ity \cdot Z$,

$$(1-\epsilon)\frac{1}{t}\frac{t|y\cdot Z|}{\sqrt{1-(x\cdot Z)^2}} \leq \frac{1}{t}\log|h((x+ity)\cdot Z)| \leq \frac{1}{t}\frac{t|y\cdot Z|}{\sqrt{1-(x\cdot Z)^2}}. \tag{3.6}$$

This gives

$$\lim_{t\to 0^+}\frac{V_K(x+ity)}{t} = \sup\{\frac{|y\cdot Z|}{\sqrt{1-(x\cdot Z)^2}} : Z \in K^*\}. \tag{3.7}$$

To relate this with $b^*(x;y)$, in the symmetric case, the $b-$maximal ellipse is easily seen to be an $a-$maximal ellipse (see the next proposition for a generalization of this), and the linear polynomial p that maps the support "strip" of this ellipse to $[-1,1]$ (i.e., it maps one parallel support hyperplane to -1 and the other to $+1$) is easily seen to give

$$\frac{1}{\deg p}\frac{|D_y p(x)|}{\sqrt{1-p(x)^2}} = \frac{1}{b^*(x,y)}$$

so that we have the equality $\delta_M(x,y) = \frac{1}{b^*(x,y)}$. Thus, the first part of the proposition is proved. Moreover, we have the following formula for $b^*(x,y)$:

$$b^*(x;y) = \inf\{\frac{\sqrt{1-(x\cdot w)^2}}{|y\cdot w|} : w\in K^*\}.$$

To see this, in the symmetric case one considers symmetric ellipses in (3.1), i.e., $a := x$, and, from the definition of $b^*(x;y)$ and K^* we can write

$$\begin{aligned} b^*(x;y) &= \sup\{b : \sup_{w\in K^*,\ t\in[0,2\pi]} |x\cos t\cdot w + yb\sin t\cdot w| = 1\} \\ &= \sup\{b : \sup_{w\in K^*} [(w\cdot x)^2 + b^2(w\cdot y)^2] = 1\}. \end{aligned}$$

Basically unwinding things shows that this is the reciprocal of (3.7). For details we refer the reader to [11] or [5].

A key geometric observation which will be used in the next section is the following.

Proposition 3.2. *For any convex body K, a $b-$maximal ellipse $\mathcal{E}$ is also an $a-$maximal ellipse.*

Proof. First observe that an $a-$maximal ellipse $\mathcal{E}$ is characterized by the property that *no translate of $\mathcal{E}$ lies entirely in the interior K^o of K*. For if $\mathcal{E}+v\subset K^o$ for some $v\neq 0$, one can dilate $\mathcal{E}+v$ to get an ellipse with the same orientation and eccentricity as $\mathcal{E}$ which lies in K but has larger area. Conversely, if $\mathcal{E}$ is not an $a-$maximal ellipse, then one can find an ellipse $\mathcal{E}'$ with the same orientation and eccentricity as $\mathcal{E}$ which lies in K but has larger area. The convex hull H of $\mathcal{E}\cup\mathcal{E}'$ lies in K and we can translate $\mathcal{E}$ within H to an ellipse $\mathcal{E}''$ lying in the two-dimensional surface $S(\mathcal{E}')$ determined by $\mathcal{E}'$; if $\mathcal{E}''$ does not lie in the "interior" of $S(\mathcal{E}')$, we simply translate it within this surface (since the area of $\mathcal{E}'$! is greater than that of $\mathcal{E}''$) until it does.

Indeed, we need a slightly more precise statement: $\mathcal{E}$ is not an $a-$maximal ellipse if and only if there is a unit vector v and $\delta > 0$ such that $\mathcal{E} + sv \subset K^o$ for $0 < s < \delta$, i.e., all translates by a small amount in some direction stay in K^o. This follows from the previous paragraph if we observe the following fact: if K is a convex body, $u \in K$ and $u + v \in \partial K^o$, then the entire half-open segment $(u, u + v]$ lies in K^o.

Suppose that $\mathcal{E}$ given by

$$\theta \to a\cos\theta + by\sin\theta + (x - a)$$

is a $b-$maximal ellipse for x, y. For the sake of obtaining a contradiction, we assume that $\mathcal{E}$ is not an $a-$maximal ellipse. By the previous paragraph, we can find a nonzero vector v and $\delta > 0$ so that $\mathcal{E}_s := \mathcal{E} + sv$ lies in K^o for $0 < s < \delta$. For $0 < \epsilon < \delta/2$, consider the ellipse $\tilde{\mathcal{E}}(\epsilon)$ given by

$$r_\epsilon(\theta) = (a - \epsilon v)\cos\theta + by\sin\theta + x - (a - \epsilon v). \tag{3.8}$$

We claim that $\tilde{\mathcal{E}}(\epsilon) \subset K^o$. Assuming this is the case, note that $r_\epsilon(0) = x \in \tilde{\mathcal{E}}(\epsilon)$ and $r'_\epsilon(0) = by$; in particular, the "b" for $\tilde{\mathcal{E}}(\epsilon)$ is the same as the "b" for $\mathcal{E}$. Since $\tilde{\mathcal{E}}(\epsilon) \subset K^o$, we can modify $\tilde{\mathcal{E}}(\epsilon)$ to an ellipse $\tilde{\mathcal{E}}(\epsilon)'$ containing x and lying in K by replacing b in (3.8) by $b' > b$ contradicting the assumption that $\mathcal{E}$ is a $b-$maximal ellipse for x, y.

To verify that $\tilde{\mathcal{E}}(\epsilon) \subset K^o$, observe that for each fixed θ, the point

$$(a - \epsilon v)\cos\theta + by\sin\theta + x - (a - \epsilon v)$$

$$= a\cos\theta + by\sin\theta + (x - a) + \epsilon v(1 - \cos\theta)$$

on $\tilde{\mathcal{E}}(\epsilon)$ lies on the ellipse $\mathcal{E}_{s_\theta} := \mathcal{E} + \epsilon(1 - \cos\theta)v$ where $s_\theta = \epsilon(1 - \cos\theta) \leq 2\epsilon < \delta$. Thus $\tilde{\mathcal{E}}(\epsilon) \subset K^o$.

□

For use in the next section, we prove some results about the function $b^*(x, y)$.

Proposition 3.3. *For a convex body $K \subset \mathbb{R}^n$, $b^*(x, y)$ defined in (3.3) is a continuous function of $x \in K^o$ and $y \in \mathbb{R}^n$. Moreover, for each fixed $x \in K^o$, $y \to 1/b^*(x, y)$ is a norm in $\mathbb{R}^n$.*

Proof. For the continuity of $b^*(x, y)$, we first verify uppersemicontinuity of this function. Fix a convex body K and fix $x \in K^o$ and $y \in \mathbb{R}^n$. Let

$\{x_j\} \subset K^o$ with $x_j \to x$ and $\{y_j\} \subset \mathbb{R}^n$ with $y_j \to y$. Let

$$r_j(\theta) = a_j \cos\theta + b^*(x_j, y_j)y_j \sin\theta + (x_j - a_j)$$

parameterize a $b-$maximal ellipse $\mathcal{E}_j$ for K through x_j in the direction y_j. Take a subsequence $\{j_k\}$ of positive integers so that the numbers $\{b^*(x_{j_k}, y_{j_k})\}$ converge to a number $\tilde{b}$; and take a further subsequence (which we still call $\{j_k\}$) so that the vectors $\{a_{j_k}\} \subset \mathbb{R}^n$ converge to $a \in \mathbb{R}^n$. Consider the ellipse $\mathcal{E}$ where

$$r(\theta) = a\cos\theta + \tilde{b}y\sin\theta + (x - a).$$

Since $x_{j_k} \to x,\ y_{j_k} \to y,\ b^*(x_{j_k}, y_{j_k}) \to \tilde{b}$ and $a_{j_k} \to a$, the functions r_{j_k} converge uniformly to r (equivalently, the ellipses $\mathcal{E}_{j_k}$ converge in the Hausdorff metric to $\mathcal{E}$). Thus $\mathcal{E}$ is an inscribed ellipse for K through x in the direction of y; hence $\tilde{b} \le b^*(x, y)$; i.e.,

$$\limsup_{x' \to x,\ y' \to y} b^*(x', y') \le b^*(x, y).$$

To verify lowersemicontinuity of $b^*(x, y)$, we fix $x \in K^o,\ y \in \mathbb{R}^n$ and $b' < b^*(x, y)$, and we show there is a $\delta > 0$ such that for all $|x' - x| < \delta,\ |y' - y| < \delta$ there is an inscribed ellipse $\mathcal{E}'$ through x' with tangent direction y' of the form

$$\theta \to a'\cos\theta + b'y'\sin\theta + (x' - a').$$

Let $\mathcal{E}$ be a $b-$maximal ellipse through x in the direction y given by

$$r(\theta) = a\cos\theta + b^*(x, y)y\sin\theta + (x - a).$$

If $x - 2a \in K^o$, then for $b' < b$ the ellipse $\mathcal{E}_{b'}$

$$r_{b'}(\theta) = a\cos\theta + b'y\sin\theta + (x - a)$$

lies fully in K^o (for $\mathcal{E}_{b'}$ lies entirely "inside" of $\mathcal{E}$ except for the common points $x, x - 2a$, and $x \in K^o$). Then any sufficiently small translation $\mathcal{E}'$

$$\theta \to a\cos\theta + b'y\sin\theta + (x' - a)$$

of $\mathcal{E}_{b'}$ by $x' - x$ keeps $\mathcal{E}'$ in K^o; hence replacing y by y' sufficiently close to y yields $\mathcal{E}''$

$$\theta \to a\cos\theta + b'y'\sin\theta + (x' - a)$$

in K.

If $x - 2a \notin K^o$, we first modify $\mathcal{E}_{b'}$ to $\mathcal{E}_{b',a'}$:

$$r_{b',a'}(\theta) = a'\cos\theta + b'y\sin\theta + (x - a')$$

with $a' - a = \delta(a - x)$ with $\delta > 0$ sufficiently small so that $\mathcal{E}_{b',a'} \subset K^o$. This is possible since the vectors

$$r_{b',a'}(\theta) - r_{b'}(\theta) = (a - a')(1 - \cos\theta) = \delta(1 - \cos\theta)(x - a)$$

point in the same direction for all θ. Note that

$$r_{b',a'}(0) = x \text{ and } r'_{b',a'}(0) = b'y$$

so that once again any sufficiently small translation $\mathcal{E}'$

$$\theta \to a' \cos\theta + b'y \sin\theta + (x' - a')$$

of $\mathcal{E}_{b',a'}$ by $x' - x$ keeps $\mathcal{E}'$ in K^o. Again, replacing y by y' sufficiently close to y yields $\mathcal{E}''$

$$\theta \to a' \cos\theta + b'y' \sin\theta + (x' - a')$$

in K. This completes the proof that $(x, y) \to b^*(x, y)$ is continuous.

To show that $y \to 1/b^*(x, y)$ is a norm, observe that $b^*(x, \lambda y) = \frac{1}{\lambda} b^*(x, y)$ for $\lambda > 0$ and $0 < b^*(x, y) < +\infty$ if $y \neq 0 \in \mathbb{R}^n$. Thus $b^*(x, 0) = +\infty$ so that $1/b^*(x, y) \geq 0$ with equality precisely when $y = 0$; and $1/b^*(x, \lambda y) = \lambda/b^*(x, y)$ for $\lambda \geq 0$. To verify subadditivity in y, fix $x \in K^o$ and $y_1, y_2 \in \mathbb{R}^n$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $b-$maximal ellipses through x in the directions y_1 and y_2 as in (3.1):

$$\theta \to r_j(\theta) := a_j \cos\theta + b_j y_j \sin\theta + (x - a_j), \ j = 1, 2;$$

here, $b_j := b^*(x, y_j)$. Consider the convex combination

$$\begin{aligned} &\frac{b_2}{b_1 + b_2} r_1(\theta) + \frac{b_1}{b_1 + b_2} r_2(\theta) \\ &= \frac{a_1 b_2}{b_1 + b_2} \cos\theta + \frac{b_1 b_2}{b_1 + b_2} y_1 \sin\theta + \frac{b_2}{b_1 + b_2}(x - a_1) \\ &+ \frac{a_2 b_1}{b_1 + b_2} \cos\theta + \frac{b_1 b_2}{b_1 + b_2} y_2 \sin\theta + \frac{b_1}{b_1 + b_2}(x - a_2) \\ &= \frac{a_1 b_2 + a_2 b_1}{b_1 + b_2} \cos\theta + \frac{b_1 b_2}{b_1 + b_2}(y_1 + y_2) \sin\theta + x - \frac{a_1 b_2 + a_2 b_1}{b_1 + b_2}. \end{aligned}$$

By convexity, this ellipse, through x in the direction $y_1 + y_2$, lies in K so that

$$b^*(x, y_1 + y_2) \geq \frac{b_1 b_2}{b_1 + b_2}.$$

Unwinding, this says that

$$\frac{1}{b^*(x, y_1 + y_2)} \leq \frac{1}{b_1} + \frac{1}{b_2},$$

as desired. □

For future use, we mention that in $\mathbb{R}^2$, if $\mathcal{E}$ is an $a-$maximal ellipse for K, then either

(1) $\mathcal{E} \cap \partial K$ contains exactly two points a_1, a_2, in which case the tangent lines to $\mathcal{E}$ at a_1, a_2 are parallel and determine a strip S containing K and $\mathcal{E}$ is an $a-$maximal ellipse for any rectangular truncation T of S with $\mathcal{E} \subset K \subset T$; or

(2) $\mathcal{E} \cap \partial K$ contains $m \geq 3$ points $a_1, ..., a_m$, in which case either a subset of three points from $\{a_1, ..., a_m\}$ can be found so that the tangent lines to $\mathcal{E}$ at these three points bound a triangle T containing K and $\mathcal{E}$ is an $a-$maximal ellipse for T, or a rectangular truncation T of a strip S with $\mathcal{E} \subset K \subset T$ can be found so that $\mathcal{E}$ is an $a-$maximal ellipse for T.

4. Main result.

For any compact set $K \subset \mathbb{R}^n$ with non-empty interior, take $x \in K^o$ and $y \in \mathbb{R}^n \setminus \{0\}$. Then we always have the pointwise inequality

$$\delta_M(x,y) \leq \delta_B^{(i)}(x,y) := \liminf_{t \to 0^+} \frac{V_K(x+ity)}{t}. \tag{4.1}$$

This follows from Proposition 2.1 in [5]. In particular, this inequality holds for any convex body K, with equality in case K is symmetric (as we saw in the previous section). In this section, we prove that the limit $\lim_{t\to 0^+} \frac{V_K(x+ity)}{t}$ exists and equals $1/b^*(x,y)$. This verifies "Hypothesis A" in [12] for convex bodies $K \subset \mathbb{R}^n$, i.e., the inscribed ellipse method and the pluripotential-theoretic method for obtaining Bernstein-Markov-type estimates for convex bodies are equivalent.

Let K be an arbitrary convex body in $\mathbb{R}^n$. Fix $x \in K^o$ and $y \in \mathbb{R}^n \setminus \{0\}$. Take a $b-$maximal ellipse $\mathcal{E}$ through x with tangent direction y at x. We will normalize and assume that y is a unit vector; moreover, it will be convenient to have the center at a instead of $x-a$. Thus we write

$$\theta \to r(\theta) = (x-a)\cos\theta + b^*(x,y)y\sin\theta + a, \;\; \theta \in [0, 2\pi] \tag{4.2}$$

This is an $a-$maximal ellipse $\mathcal{E}$ by Proposition 3.2; i.e., $\mathcal{E}$ forms the real points of a leaf L

$$f(\zeta) = (x-a)[\frac{1}{2}(\zeta + 1/\zeta)] + b^*(x,y)y[\frac{i}{2}(\zeta - 1/\zeta)] + a, \;\; |\zeta| \geq 1 \tag{4.3}$$

of our foliation for the extremal function V_K. We can compare this "$b-$maximal" form of the leaf with its $a-$maximal form (2.2):

$$f(\zeta) = A + c\zeta + \overline{c}/\zeta,\ |\zeta| \geq 1, \tag{4.4}$$

where, for simplicity, we write $c := \rho C$ in (2.2). Thus, from (2.3), $V_K(f(\zeta)) = \log|\zeta|$ for $|\zeta| \geq 1$.

In these coordinates $V_K(f(\zeta)) = \log|\zeta|$. We first show that

$$\lim_{r\to 1^+} \frac{f(r) - f(1)}{r-1} = ib^*(x,y)y.$$

This follows from the calculation

$$f(r) - f(1) = (x-a)\Big(\frac{(r-1)^2}{2r}\Big) + ib^*(x,y)\frac{(r-1)(r+1)}{2r}.$$

Thus the real tangent vector to the real curve $r \to f(r),\ r \geq 1$ as $r \to 1^+$ is in the direction $ib^*(x,y)y$. Now $f(1) = x$ and $x \in K$ so $V_K(f(1)) = V_K(x) = 0$; and, since f is a leaf of our foliation, $V_K(f(r)) = \log r$. Hence

$$\frac{V_K(f(r)) - V_K(f(1))}{r-1} = \frac{\log r}{r-1}$$

so that

$$\lim_{r\to 1^+} \frac{V_K(f(r)) - V_K(f(1))}{r-1} = \lim_{r\to 1^+} \frac{\log r}{r-1}$$

exists and equals 1. This elementary calculation shows that for any convex body $K \subset \mathbb{R}^n$,

$$\lim_{r\to 1^+} \frac{V_K(f(r)) - V_K(f(1))}{b^*(x,y)(r-1)} = \frac{1}{b^*(x,y)}; \tag{4.5}$$

i.e., the curvilinear limit along the curve $f(r)$ in the direction of iy at x exists and equals $\frac{1}{b^*(x,y)}$.

Note that

$$f(r) - f(1) = f(r) - x = ib^*(x,y)y(r-1) + 0((r-1)^2),$$

so that the point $x + ib^*(x,y)y(r-1)$ is $O((r-1)^2)$ close to the point $f(r)$. We use the explicit form (4.3) of the leaf to verify the existence of the limit in the directional derivative $\delta_B(x,y)$.

Theorem 4.1. *Let K be a convex body in $\mathbb{R}^n$. Then the limit in the definition of the directional derivative exists and equals $\frac{1}{b^*(x,y)}$:*

$$\delta_B(x,y) := \lim_{t\to 0^+} \frac{V_K(x+ity)}{t} = \frac{1}{b^*(x,y)}.$$

Proof. If we can show

$$\lim_{r\to 1^+}\frac{V_K(f(r))-V_K(x+ib^*(x,y)y(r-1))}{b^*(x,y)(r-1)}=0, \tag{4.6}$$

then using (4.5) and the preceeding discussion, we will have

$$\lim_{t\to 0^+}\frac{V_K(x+ity)}{t}=\frac{1}{b^*(x,y)}. \tag{4.7}$$

We first consider the case when K admits a continuous foliation; i.e., $K\in\mathcal{C}$. Consider a fixed point $w:=x+ib^*(x,y)y(r-1)\in\mathbb{C}^n$. This belongs to some foliation leaf M which we write in the form (4.4):

$$g(\zeta)=\alpha+\gamma\zeta+\overline{\gamma}/\zeta:\ \mathbb{C}\setminus\Delta\to M\subset\mathbb{C}^n.$$

We need to use the facts that when $r\to 1^+$, then $w\to x\in L$, and, by continuity of the foliation, the leaf parameters for (g,M) should converge to those of (f,L); i.e., $\alpha\to A$ and $\gamma\to c$. We remark that if we compare (4.3) and (4.4), writing $b:=b^*(x,y)$ we have the relations

$$A=a \text{ and } c=\frac{1}{2}(x-a+iby). \tag{4.8}$$

Here we supress a rotational invariance: the substitution $\zeta':=\zeta e^{i\varphi}$ for any fixed constant φ describes the same leaf with a different parametrization; thus we fix its value so that

$$\xi:=g(1)=2\mathrm{Re}\,\gamma+\alpha$$

is closest to $x:=f(1)=2\mathrm{Re}\,c+A$, i.e., $|g(1)-f(1)|\le|g(e^{i\theta})-f(1)|$ for all θ. To emphasize, writing the leaf (f,L) in $b-$maximal form (4.3),

$$f(\zeta)=(x-a)\frac{1}{2}(\zeta+\frac{1}{\zeta})+by\frac{i}{2}(\zeta-\frac{1}{\zeta})+a$$

where, from (4.8) and the fact that y is a unit vector, $b:=2|\mathrm{Im}\,c|>0$ and $y:=\frac{2}{b}\mathrm{Im}\,c\in\mathbb{R}^n$. Now, apriori, we do not know if (g,M) is $b-$maximal (aposteriori, it is: see Corollary 4.3). However, we may still write this leaf in the form

$$g(\zeta)=(\xi-\alpha)\frac{1}{2}(\zeta+\frac{1}{\zeta})+\beta\eta\frac{i}{2}(\zeta-\frac{1}{\zeta})+\alpha$$

with $\beta:=2|\mathrm{Im}\,\gamma|>0$ and $\eta:=\frac{2}{\beta}\mathrm{Im}\,\gamma\in\mathbb{R}^n$. Note that continuity of the foliation implies $\beta>0$ since $b>0$; indeed, $\beta\sim b$, $\xi\sim x$, $\eta\sim y$, $\alpha\sim a$, and $\gamma\sim c$.

Since $w \in M$, there is a point $\omega \in \mathbb{C} \setminus \Delta$ with $g(\omega) = w$. Our task is to calculate $V_K(w) = V_K(g(\omega))$. On a leaf of the foliation we have the formula $V_K(g(\omega)) = \log|\omega|$, so it suffices to compute $\log|\omega|$. The representation of w as $g(\omega)$ means that for $j = 1, \ldots, n$,

$$x_j + iby_j(r-1) = w_j = g_j(\omega) = (\xi_j - \alpha_j)\frac{1}{2}(\omega + \frac{1}{\omega}) + \beta\eta_j\frac{i}{2}(\omega - \frac{1}{\omega}) + \alpha_j.$$

Since y and η are unit vectors which are close to each other, we can choose a coordinate j with $y_j \neq 0$, $\eta_j \neq 0$. For this coordinate j, the previous displayed equation gives

$$\frac{1}{2}(\xi_j - \alpha_j + i\beta\eta_j)\omega^2 + (\alpha_j - x_j - iby_j(r-1))\omega + \frac{1}{2}(\xi_j - \alpha_j - i\beta\eta_j) = 0,$$

a quadratic equation in ω. Corresponding to the double mapping properties of the Joukowski map $\frac{1}{2}(\zeta + 1/\zeta)$, there are two roots, one in $|\zeta| < 1$ and one in $|\zeta| > 1$, the latter being our ω as we considered the mapping of the exterior of the unit disc Δ. For convenience, put $\rho := b(r-1)y_j$. Since $by_j \neq 0$, $\rho \asymp r - 1$. By the quadratic formula,

$$\omega_{1,2} = \frac{x_j - \alpha_j + i\rho \pm \sqrt{(\alpha_j - x_j - i\rho)^2 - (\xi_j - \alpha_j)^2 - \beta^2\eta_j^2}}{\xi_j - \alpha_j + i\beta\eta_j}.$$

Set $Q := \beta^2\eta_j^2 + (\xi_j - \alpha_j)^2 - (x_j - \alpha_j)^2 \sim b^2y_j^2 > 0$ by continuity of the leaf parameters and choice of j. Using this and the simple formula $\sqrt{A + 2B} = \sqrt{A} + B/\sqrt{A} + O(B^2/A^{3/2})$, valid uniformly for $|B| < A/3$, say, we can rewrite the square root as

$$\begin{aligned}
&\pm \sqrt{(x_j - \alpha_j)^2 - i2(\alpha_j - x_j)\rho - \rho^2 - (\xi_j - \alpha_j)^2 - \beta^2\eta_j^2} \\
&\quad = \pm i\sqrt{Q + 2i(\alpha_j - x_j)\rho + \rho^2} \\
&\quad = \pm i\left\{\sqrt{Q} + i\frac{(\alpha_j - x_j)\rho}{\sqrt{Q}} + O(\rho^2)\right\} \\
&\quad = \pm\left\{\frac{(x_j - \alpha_j)\rho}{\sqrt{Q}} + i\sqrt{Q} + O(\rho^2)\right\}.
\end{aligned}$$

Put $P := \xi_j - \alpha_j + i\beta\eta_j$. Then

$$\begin{aligned}
|\omega_{1,2}P|^2 &= \big|(x_j-\alpha_j)(1\pm\frac{\rho}{\sqrt{Q}}) + i\big[\pm\sqrt{Q}+\rho\big] + O(\rho^2)\big|^2 \\
&= \big[(\alpha_j - x_j) \pm \frac{\rho(\alpha_j - x_j)}{\sqrt{Q}}\big]^2 + \big[\pm\sqrt{Q}+\rho\big]^2 + O(\rho^2) \\
&= (\alpha_j - x_j)^2 \pm \frac{2\rho(\alpha_j - x_j)^2}{\sqrt{Q}} + Q \pm 2\rho\sqrt{Q} + O(\rho^2) \\
&= (\alpha_j - x_j)^2 + Q \pm \frac{2\rho}{\sqrt{Q}}\big(Q + (\alpha_j - x_j)^2\big) + O(\rho^2).
\end{aligned}$$

We have the identity $|P|^2 = (\alpha_j - x_j)^2 + Q$; dividing by this quantity on both sides yields

$$|\omega_{1,2}|^2 = 1 \pm \frac{2\rho}{\sqrt{Q}} + O(\rho^2).$$

Fixing the branch of the square-root with $\sqrt{Q} > 0$, it is clear that among the choice of signs in $\pm$ the one equal to the sign of y_j leads to the larger absolute value (and the one with $|\omega|$ exceeding 1); hence for such ω with $g(\omega) = w$,

$$|\omega|^2 = 1 + \frac{2|\rho|}{\sqrt{Q}} + O(\rho^2) = 1 + \frac{2b|y_j|(r-1)}{\sqrt{Q}} + O((r-1)^2),$$

so that

$$\log|\omega|^2 = \log\big|1 + \frac{2b|y_j|(r-1)}{\sqrt{Q}} + O((r-1)^2)\big| = \frac{2b|y_j|(r-1)}{\sqrt{Q}} + O((r-1)^2).$$

Hence

$$\frac{V_K(x+iby(r-1))}{r-1} = \frac{V_K(g(\omega))}{r-1} = \frac{\log|\omega|}{r-1} = \frac{1}{2}\frac{\log|\omega|^2}{r-1} = \frac{b|y_j|}{\sqrt{Q}} + O(r-1).$$

Recall once again that the continuity of the foliation, as $r \to 1$ we have $\xi_j \to x_j$, $\alpha_j \to a_j$, $\beta \to b$, and thus $\sqrt{Q} \to b|y_j|$. Hence

$$\lim_{r\to 1^+} \frac{V_K(x+iby(r-1))}{r-1} = 1 = \lim_{r\to 1^+} \frac{V_K(f(r))}{r-1}.$$

This verifies (4.6) and hence (4.7) in the case when K admits a continuous foliation.

For a general convex body K, we use the previous case and an appropriate approximation argument to verify (4.7). To emphasize the set(s) under discussion, we write

$$b^*(K;x,y) := b^*(x,y) \text{ for the set } K.$$

We need to prove two inequalities:

$$\frac{1}{b^*(K;x,y)} \le \liminf_{t\to 0+} \frac{V_K(x+ity)}{t} \tag{4.9}$$

and

$$\frac{1}{b^*(K;x,y)} \ge \limsup_{t\to 0+} \frac{V_K(x+ity)}{t}. \tag{4.10}$$

Note that if K and κ are two convex bodies, we have for any $y \in \mathbb{R}^n$ the inequalities

$$b^*(K;x,y) \begin{cases} \le b^*(\kappa;x,y) & \text{if} \quad K \subset \kappa \\ \ge b^*(\kappa;x,y) & \text{if} \quad \kappa \subset K \end{cases}, \tag{4.11}$$

and for any $y \in \mathbb{R}^n$ and any $t \in \mathbb{R}$ the inequalities

$$V_K(x+ity) \begin{cases} \ge V_\kappa(x+ity) & \text{if} \quad K \subset \kappa \\ \le V_\kappa(x+ity) & \text{if} \quad \kappa \subset K \end{cases}. \tag{4.12}$$

Fix $\alpha < 1$ arbitrarily close to 1 and choose $\delta > 0$ small (to be determined later in terms of α). From the discussion at the end of section 2, $\mathcal{C}$ is dense in $\mathcal{K}$; thus we can find $\kappa \in \mathcal{C}$ such that the Hausdorff distance between κ and K is at most δ. Take an α-dilated (at x) copy K_1 of K and an α-dilated copy κ_1 of κ. Then $x \in K_1^0$, and if $\delta = \delta(\alpha)$ is sufficiently small, we have $\kappa_1 \subset K$. We also take the $1/\alpha$-dilated copies K_2 and κ_2 of K and κ. Again for small enough δ, we will have $K \subset \kappa_2$. Note that κ_2 is the α^{-2}-dilated copy of κ_1; hence $b^*(\kappa_2;x,y) = \alpha^{-2} b^*(\kappa_1;x,y)$. Therefore, using (4.7) for κ_1 and κ_2, we obtain

$$\lim_{t\to 0+} \frac{V_{\kappa_2}(x+ity)}{t} = \frac{1}{b^*(\kappa_2;x,y)} = \alpha^2 \frac{1}{b^*(\kappa_1;x,y)} = \alpha^2 \lim_{t\to 0+} \frac{V_{\kappa_1}(x+ity)}{t}. \tag{4.13}$$

Using (4.13), (4.11) and (4.12), we obtain

$$\begin{aligned}\frac{1}{b^*(K;x,y)} \leq \frac{1}{b^*(\kappa_1;x,y)} &= \lim_{t\to 0+} \frac{V_{\kappa_1}(x+ity)}{t} \\ &= \frac{1}{\alpha^2}\lim_{t\to 0+} \frac{V_{\kappa_2}(x+ity)}{t} = \frac{1}{\alpha^2}\liminf_{t\to 0+} \frac{V_{\kappa_2}(x+ity)}{t} \\ &\leq \frac{1}{\alpha^2}\liminf_{t\to 0+} \frac{V_K(x+ity)}{t}; \qquad (4.14)\end{aligned}$$

and, in a similar fashion we get

$$\begin{aligned}\frac{1}{b^*(K;x,y)} \geq \frac{1}{b^*(\kappa_2;x,y)} &= \lim_{t\to 0+} \frac{V_{\kappa_2}(x+ity)}{t} \\ &= \alpha^2\lim_{t\to 0+} \frac{V_{\kappa_1}(x+ity)}{t} = \alpha^2\limsup_{t\to 0+} \frac{V_{\kappa_1}(x+ity)}{t} \\ &\geq \alpha^2\limsup_{t\to 0+} \frac{V_K(x+ity)}{t}. \qquad (4.15)\end{aligned}$$

Since α can be taken arbitrarily close to 1, (4.9) and (4.10) follow from (4.14) and (4.15).

□

Remark 4.2. Observe that the essential property used to verify (4.6) for $K \in \mathcal{C}$ is (2.3); i.e., that $V_K(f(\zeta)) = \log|\zeta|$ on L (i.e., for $|\zeta| \geq 1$), which is equivalent to the $a-$maximality of the real ellipse $\mathcal{E} \subset L$ (see section 2).

Corollary 4.3. *For any convex body K, an ellipse $\mathcal{E} \subset K$ is $a-$maximal if and only if it is $b-$maximal for all $x \in K^o$ and $y \in T_x\mathcal{E}$.*

Proof. That $b-$maximality implies $a-$maximality was proved in Proposition 3.2. For the converse, we first suppose that $K \in \mathcal{C}$. Let $\mathcal{E}$ be an $a-$maximal ellipse. Fix $x \in K^o$ and $y \in T_x\mathcal{E}$ a unit vector. Then $\mathcal{E} = f(\partial\Delta)$ where

$$f(\zeta) = (x-\alpha)\frac{1}{2}(\zeta + \frac{1}{\zeta}) + \beta y\frac{i}{2}(\zeta - \frac{1}{\zeta}) + \alpha \qquad (4.16)$$

and $V_K(f(\zeta)) = \log|\zeta|$ for $|\zeta| \geq 1$, so that, from the remark,

$$1/\beta = \lim_{r\to 1^+} \frac{V_K(f(r))}{\beta(r-1)} = \lim_{r\to 1^+} \frac{V_K(x+i\beta y(r-1))}{\beta(r-1)}.$$

Writing $t = \beta(r-1)$ in the limit on the right and using Theorem 4.1,

$$1/\beta = \lim_{t\to 0^+} \frac{V_K(x+ity)}{t} = \frac{1}{b^*(x,y)}$$

so that $\mathcal{E}$ is $b-$maximal for x, y.

Now let $K \in \mathcal{K}$ be an arbitrary convex body. We consider first the case where $\mathcal{E}$ is the unique $a-$maximal ellipse for $[c] \in H$; i.e., for its orientation and eccentricity, and we again write $\mathcal{E} = f(\partial\Delta)$ as in (4.16). Take a sequence $\{K_j\} \subset \mathcal{C}$ with $K_j \searrow K$. For each j, let $\mathcal{E}_j$ be the unique $a-$maximal ellipse for $K_j, [c]$, and let f_j denote the corresponding leaf. Then (cf., [7]) $f_j \to f$ uniformly so that $\mathcal{E}_j \to \mathcal{E}$. As in the proof of Theorem 4.1, we may write

$$f_j(\zeta) = (x_j - \alpha_j)\frac{1}{2}(\zeta + \frac{1}{\zeta}) + \beta_j y_j \frac{i}{2}(\zeta - \frac{1}{\zeta}) + \alpha_j.$$

By the first part of the proof, $\mathcal{E}_j$ is b-maximal in K_j, so that $\beta_j = b^*(x_j, y_j, K_j)$. The uniform convergence of f_j to f implies that $\alpha_j \to \alpha$, $x_j \to x$ and $y_j \to y$. Moreover, since $x \in K^o$, for j sufficiently large, $x_j \in K^o$. From the continuity of b^* (Proposition 3.3) and the fact that $K \subset K_j$,

$$b^*(x,y,K) = \lim_{j\to\infty} b^*(x_j,y_j,K) \leq \lim_{j\to\infty} b^*(x_j,y_j,K_j) = \lim_{j\to\infty}\beta_j = \beta.$$

Hence $\beta = b^*(x,y,K)$ and $\mathcal{E}$ is $b-$maximal.

In the case where $\mathcal{E}$ is not the unique $a-$maximal ellipse for the corresponding $[c] \in H$, it is an $a-$maximal ellipse for this $[c]$ and a "strip" S, i.e., a closed body S bounded by two parallel hyperplanes P_1, P_2 with $K \subset S$ (see section 7 of [7]). If $\mathcal{E}$ is given by

$$r(\theta) = f(e^{i\theta}) = (x-\alpha)\cos\theta + \beta y \sin\theta + \alpha$$

then there is θ_0 such that $r(\theta_0) \in P_1$ and $r(\theta_0 + \pi) \in P_2$. It is therefore sufficient to show that any ellipse $\mathcal{E} \subset K$ that intersects P_1, P_2 is b-maximal for any $x \in \mathcal{E} \cap K^o$, $y \in T_x\mathcal{E}$. Take a sufficiently large convex set T that is symmetric about the center of the ellipse (e.g., a large box) so that $K \subset T \subset S$. Clearly $\mathcal{E}$ is a-maximal for T. Since T is symmetric, $T \in \mathcal{C}$, so by the first part of the proof, $\mathcal{E}$ is b-maximal for T, x, y. Hence it is also b-maximal for K, x, y. □

We turn to the Monge-Ampere measure. We know that $(dd^cV_K)^n$ is supported in K and is absolutely continuous with respect to Lebesgue

measure dx on K^o, i.e., $(dd^cV_K)^n = \tilde{\lambda}(x)dx$ for a locally integrable nonnegative function $\tilde{c}$ on K^o [3]. Baran has proved the following (see [2], Propositions 1.10, 1.11 and Lemma 1.12).

Proposition 4.4. *Let $D \subset \mathbf{C}^n$ and let $\Omega := D \cap \mathbb{R}^n$. Let u be a nonnegative psh function on D which satisfies:*

i. $\Omega = \{u = 0\}$
ii. $(dd^cu)^n = 0$ on $D \setminus \Omega$
iii. $(dd^cu)^n = \lambda(x)dx$ *on* Ω *where* $c \in L^1_{loc}(\Omega)$
iv. *for all* $x \in \Omega,\ y \in \mathbb{R}^n$, *the limit*

$$h(x, y) := \lim_{t \to 0^+} \frac{u(x + ity)}{t} \text{ exists and is continuous on } \Omega \times i\mathbb{R}^n$$

v. $x \in \Omega, y \to h(x, y)$ *is a norm.*

Then

$$\lambda(x) = n!\mathrm{vol}\{y : h(x, y) \leq 1\}^*$$

and $\lambda(x)$ is a continuous function on Ω.

We now obtain the generalization of (1.3).

Corollary 4.5. *Let K be a convex body in $\mathbb{R}^n$. Then*

$$(dd^cV_K)^n = \lambda(x)dx \text{ for } x \in K^o$$

where $\lambda(x) = n!\mathrm{vol}(\{y : \delta_B(x, y) = \frac{1}{b^*(x,y)} \leq 1\}^*)$ is continuous. Moreover, $(dd^cV_K)^n$ puts no mass on the boundary ∂K (relative to $\mathbb{R}^n$).

Proof. The formula for $(dd^cV_K)^n$ on K^o is immediate from Theorem 4.1, Proposition 4.4 and the paragraph preceeding it, and Proposition 3.3. To show that $(dd^cV_K)^n$ puts no mass on the boundary ∂K, we proceed as in [3]. Let $\{K_j\}$ be a sequence of convex bodies in $\mathbb{R}^n$ with real-analytic boundaries $\{\partial K_j\}$ such that K_j increases to K. Then ∂K_j is pluripolar so that $(dd^cV_{K_j})^n$ puts no mass on ∂K_j (cf., Proposition 4.6.4 of [8]). Writing $(dd^cV_{K_j})^n := \lambda_j(x)dx$ for $x \in K_j^o$, we have

$$(2\pi)^n = \int_{K_j^o} \lambda_j(x)dx = \int_{K^o} \lambda_j(x)dx$$

where we extend $\lambda_j(x)$ to be zero outside of K_j^o. Since $\int_{K^o} \lambda_j(x)dx \leq (2\pi)^n < \infty$, and the density functions $\lambda_j(x)$ increase almost everywhere on K^o to $\lambda(x)$ (cf., (4.11)), by dominated convergence we have

$$(2\pi)^n = \lim_{j \to \infty} \int_{K^o} \lambda_j(x)dx = \int_{K^o} \lambda(x)dx.$$

Thus $(dd^cV_K)^n$ puts no mass on the boundary ∂K. $\square$

We end this note with a final remark on Bernstein-Markov inequalities. Baran [2] conjectured that we have equality $\delta_M(x,y) = \delta_B^{(i)}(x,y)$ in (4.1) for general convex bodies. With respect to this conjecture, we make the following observation: *if we can prove $\delta_M(x,y) = \delta_B(x,y)$ for a triangle T in $\mathbb{R}^2$, then equality holds for all convex bodies in $\mathbb{R}^2$.* For let K be a convex body in $\mathbb{R}^2$. Fix $x \in K^o$ and $|y| = 1$. Take a $b-$maximal ellipse $\mathcal{E} = \mathcal{E}(x,y)$ for K with parameter $b = b^*(x,y)$ and, as in (1) or (2) at the end of section 3, take a rectangle or triangle T containing K in which $\mathcal{E}$ is an $a-$maximal ellipse. Then since $\delta_B^T(x,y) = \frac{1}{b^*(x,y)} = \delta_B^K(x,y)$, we have

$$\delta_M^K(x,y) \geq \delta_M^T(x,y) = \delta_B^T(x,y) = \delta_B^K(x,y).$$

From (4.1), $\delta_M^K(x,y) \leq \delta_B^K(x,y)$ and equality holds.

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